WADD TECHNICAL REPORT 60-902

POLYNOMIAL EXPANSIONS OF BESSEL FUNCTIONS AND SOME ASSOCIATED FUNCTIONS

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Midwest Research Institute

April 1961

Directorate of Systems Engineering
Contract No. AF 33(616)-6535
Project No. 7062
Task No. 70953

AERONAUTICAL SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

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FOREWORD

This work was supported by the United States Air Force through the Wright Air Development Division (now Aeronautical Systems Division) under Contract No. AF 33(616)-6535. The contract was initiated under Task No. 70924, "Mathematical Research in Numerical Methods of Computing Higher Transcendental Functions."

This report covers work conducted from October 1959 to July 1960.

A summary of other research conducted in this program is described in "Polynomial Approximations to Integral Transforms" by J. Jet Wimp, WADC Technical Note 59-314, September 1959.

The author wishes to extend his grateful thanks to Yudell L. Luke for his many helpful comments and suggestions during the preparation of this report.

ABSTRACT

In this report we first determine representations for the Anger-Weber functions $\mathbb{J}_\nu(ax)$ and $\mathbb{E}_\nu(ax)$ in series of symmetric Jacobi polynomials. (These include Legendre and Chebyshev polynomials as special cases.) If ν is an integer, these become expansions for the Bessel function of the first kind, since $\mathbb{J}_n(ax) = J_n(ax)$. Next, corresponding representations are found for $(ax)^{-\nu}J_\nu(ax)$. Convenient error bounds are obtained for the Chebyshev cases of the above expansions.

In the final section of the report we determine the similar type expansions for the Bessel functions $Y_n(ax)$ and $K_n(ax)$.

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1. Symmetric Jacobi Expansions of Anger-Weber Functions

A function $f(x)$ satisfying certain conditions (for these consult [1]*) may be expanded in the series

$$f(x) = \sum_{n=0}^{\infty} C_n P_n^{(\alpha,\alpha)}(x) \ , \ -1 \leq x \leq 1 \ , \ \alpha > -1 \quad , \tag{1.1}$$

where $P_n^{(\alpha,\alpha)}(x)$ is called the symmetric Jacobi polynomial of degree n. For our present purposes we shall use a definition given in [2]:

$$2^n n! P_n^{(\alpha,\alpha)}(x) = (-)^n (1-x^2)^{-\alpha} D^n \left[(1-x^2)^{\alpha+n}\right] \quad . \tag{1.2}$$

Also

$$C_n = h_n^{-1} \int_{-1}^{1} f(x)(1-x^2)^{\alpha} P_n^{(\alpha,\alpha)}(x)dx \quad , \tag{1.3}$$

$$h_n = \frac{2^{2\alpha}\Gamma(n+\alpha+1)}{n!\left(n+\alpha+\frac{1}{2}\right)(n+\alpha+1)_\alpha} \quad ; \quad (\nu)_\mu = \frac{\Gamma(\nu+\mu)}{\Gamma(\nu)} \ , \ \nu_0 = 1 \quad . \tag{1.4}$$

Using the representation (1.2) in (1.3) and noticing that all derivatives of $(1-x^2)^{\alpha+n}$ up to and including the $(n-1)$st vanish at $x = \pm 1$, we integrate (1.3) n times by parts to get:

$$C_n = (2^n n! h_n)^{-1} \int_{-1}^{1} f^{(n)}(x)(1-x^2)^{\alpha+n}dx \quad . \tag{1.5}$$

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* Numbers in brackets refer to bibliography at end of report.

This report was released by the author on 13 December 1960 for publication as a WADD Technical Report.

Consider the integral definition of the Anger-Weber functions [3]

$$\mathbb{J}_{\nu}(ax) + i\mathbb{E}_{\nu}(ax) = \pi^{-1}\int_0^{\pi} e^{i[\nu\phi - ax\sin\phi]}d\phi = f(x) \quad . \qquad (1.6)$$

When ν is an integer $\mathbb{J}_{\nu}(ax)$ coincides with the Bessel function of the first kind $J_{\nu}(ax)$ [4].

Now differentiate (1.6) n times under the integral sign, substitute the result in (1.5) and interchange the order of integration (which is, of course, permissible). The inner integral is known [5] and after evaluating it we have

$$C_n = (-1)^n(n+1)_{\alpha}\left(h_n\pi^{\frac{1}{2}}\right)^{-1}\int_0^{\pi} e^{i\nu\phi}\left\{\frac{a\sin\phi}{2}\right\}^{-\left(\alpha+\frac{1}{2}\right)}$$

$$\times J_{n+\alpha+\frac{1}{2}}(a\sin\phi)d\phi \quad . \qquad (1.7)$$

Use the power series expansion for the Bessel function in (1.7) and integrate term by term to get

$$C_n = (-1)^n\left[\cos\frac{\nu\pi}{2} + i\sin\frac{\nu\pi}{2}\right]\Lambda_n R_n(\nu,\alpha,a) \quad , \qquad (1.8)$$

where

$$\Lambda_n = \frac{a^n n!}{\Gamma\left(\frac{n}{2}+\frac{\nu}{2}+1\right)\Gamma\left(\frac{n}{2}-\frac{\nu}{2}+1\right)(n+2\alpha+1)_n} \quad , \qquad (1.9)$$

and R_n is conveniently described in hypergeometric notation [6] as

$$R_n(\nu,\alpha,a) = {}_2F_3\left[\frac{n}{2}+\frac{1}{2}, \frac{n}{2}+1; \alpha+n+\frac{3}{2}, \frac{n}{2}+\frac{\nu}{2}+1, \frac{n}{2}-\frac{\nu}{2}+1; -\frac{a^2}{4}\right] \quad (1.10)$$

Equating real and imaginary parts of (1.6) and (1.2) we get

$$\mathbb{J}_\nu(ax) = \sum_{n=0}^{\infty} A_n P_n^{(\alpha,\alpha)}(x), \quad -1\le x\le 1, \quad (1.11)$$

$$\mathbb{E}_\nu(ax) = \sum_{n=0}^{\infty} B_n P_n^{(\alpha,\alpha)}(x), \quad -1\le x\le 1, \quad (1.12)$$

where

$$A_n = \Lambda_n R_n(\nu,\alpha,a)\phi_n(\nu), \quad (1.13)$$

$$B_n = \Lambda_n R_n(\nu,\alpha,a)\psi_n(\nu), \quad (1.14)$$

and

$$\phi_n(\nu) = \begin{cases} (-)^{\frac{n}{2}} \cos\frac{\nu\pi}{2}, & n \text{ even}, \\ (-)^{\frac{n-1}{2}} \sin\frac{\nu\pi}{2}, & n \text{ odd}; \end{cases} \quad (1.15)$$

$$\psi_n(\nu) = \begin{cases} (-)^{\frac{n}{2}} \sin\frac{\nu\pi}{2}, & n \text{ even}, \\ (-)^{\frac{n+1}{2}} \cos\frac{\nu\pi}{2}, & n \text{ odd}. \end{cases} \quad (1.16)$$

When $\alpha = -\frac{1}{2}$,

$$P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) = \Gamma\left(n+\frac{1}{2}\right)\left(n!\,\pi^{\frac{1}{2}}\right)^{-1} T_n(x) \ , \ n = 1,2\ldots \ , \qquad (1.17)$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind of degree n . Also for this value of α, R_n simplifies to the products of two Bessel functions [7]. With $\alpha = -\frac{1}{2}$, then (1.11)-(1.14) become

$$\mathbf{J}_\nu(ax) = \sum_{n=0}^{\infty} C_n T_n(x) \ , \quad -1 \leq x \leq 1 \quad , \qquad (1.18)$$

$$\mathbf{E}_\nu(ax) = \sum_{n=0}^{\infty} D_n T_n(x) \ , \quad -1 \leq x \leq 1 \quad , \qquad (1.19)$$

where

$$C_n = \epsilon_n J_{\frac{n+\nu}{2}}\left(\frac{a}{2}\right) \ J_{\frac{n-\nu}{2}}\left(\frac{a}{2}\right) \ \phi_n(\nu) \quad , \qquad (1.20)$$

$$D_n = \epsilon_n J_{\frac{n+\nu}{2}}\left(\frac{a}{2}\right) \ J_{\frac{n-\nu}{2}}\left(\frac{a}{2}\right) \ \psi_n(\nu) \quad , \qquad (1.21)$$

and $\epsilon_n = \left[1, n=0; 2, n>0\right]$.

For integral ν we have the expansions

$$J_{2k}(ax) = \sum_{n=0}^{\infty} \epsilon_n J_{k+n}\left(\frac{a}{2}\right) J_{k-n}\left(\frac{a}{2}\right) T_{2n}(x) \ , \quad -1 \leq x \leq 1 \quad , \qquad (1.22)$$

$$J_{2k+1}(ax) = \sum_{n=0}^{\infty} 2J_{k+n+1}\left(\frac{a}{2}\right) J_{k-n}\left(\frac{a}{2}\right) T_{2n+1}(x) \ , \ -1 \leq x \leq 1 \qquad (1.23)$$

and $k = 0,1,2 \ldots$.

Since

$$J_{\nu}(iz) = e^{\frac{\nu \pi i}{2}} I_{\nu}(z) \qquad (1.24)$$

where $I_{\nu}(z)$ is the modified Bessel function of the first kind [8], we may replace a by ia in (1.22) and (1.23) to get expansions for $I_{2k}(ax)$ and $I_{2k+1}(ax)$.

It is important to note that, although the above expansions are valid only for x real and $|x| \leq 1$, (1.6) is entire in a and ν , and hence a may be chosen arbitrarily to yield expansions valid everywhere in the finite complex plane.

The expansions (1.11), (1.12), (1.18), (1.19), (1.22) and (1.23) are quite rapidly convergent, particularly in the Chebyshev cases, and consequently the last four expansions are eminently suitable for use in digital computers. Such series are usually truncated and rearranged in powers of x , although this is not necessary since $T_n(x)$ satisfies simple recursion relationships [9]*.

Let $|\epsilon_N|$ denote the maximum error incurred for $-1 \leq x \leq 1$ by taking just N terms of any of the expansions (1.22) and (1.23). Using [13] and the inequalities

$$|T_n(x)| \leq 1 \ , \quad -1 \leq x \leq 1 \quad , \qquad (1.25)$$

$$(m+n)! \geq m!n! \max\left[m^n, n^m\right] , \ m,n \ \text{integers not both zero} \ , \qquad (1.26)$$

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* The Bessel functions required to compute the coefficients in our expansions can be systematically generated on electronic computers with the aid of techniques discussed in [10, 11, 12]. There are numerous tables available for hand calculations.

we may derive error bounds for $N > k$. For (1.22),

$$|\epsilon_N| \le \frac{|a|^{2N} \exp|2\mathcal{L}\left(\frac{a}{2}\right)|}{2^{4N-1}(N+k)!\,(N-k)!}\, I_0\left(\left|\frac{a}{2\sqrt{N^2-k^2}}\right|\right) \quad . \tag{1.27}$$

and for (1.23)

$$|\epsilon_N| \le \frac{|a|^{2N+1} \exp|2\mathcal{L}\left(\frac{a}{2}\right)|}{2^{4N+1}(N+k+1)!\,(N-k)!}\, I_0\left|\frac{a}{2\sqrt{(N+k+1)(N-k)}}\right| \quad . \tag{1.28}$$

For the cases $I_{2k}(ax)$ and $I_{2k+1}(ax)$, respectively, $\exp\left|2\mathcal{L}\left(\frac{a}{2}\right)\right|$ in (1.27) and (1.28) is replaced by $\exp\left|2\mathcal{L}\left(\frac{ia}{2}\right)\right|$.

2. Expansions of Bessel Functions of the First Kind of Nonintegral Order

Results in the previous section gave symmetric Jacobi polynomial expansions for $J_\nu(ax)$ and $I_\nu(ax)$ for integral ν . When ν is non-integral, these functions are no longer entire functions of x , and it is convenient to study the entire function

$$\Gamma(\nu+1)(ax/2)^{-\nu} J_\nu(ax) = {}_0F_1\left(\nu+1; -\frac{a^2x^2}{4}\right) \quad . \tag{2.1}$$

We will derive expansions for the above ${}_0F_1$. Corresponding expansions for $\Gamma(\nu+1)(ax/2)^{-\nu} I_\nu(ax)$ then follow, as before, from (1.24).

We shall first need the following identity.

$$P_{2n}^{(\alpha,\alpha)}(x) = \binom{2n+\alpha}{2n}\binom{n+\alpha}{n}^{-1} P_n^{\left(\alpha,-\frac{1}{2}\right)}(2x^2-1) \tag{2.2}$$

where the generalized Jacobi polynomial of degree n, $P_n^{(\alpha,\beta)}(x)$ may be defined as

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} \, {}_2F_1\left[-n, n+\alpha+\beta+1; \alpha+1; \frac{1}{2} - \frac{1}{2}x\right] . \qquad (2.3)$$

The following quadratic transformation of the Gaussian hypergeometric function is given in [14]:

$$ {}_2F_1\left[2a, 2b; a+b+\frac{1}{2}; \frac{1}{2} - \frac{1}{2}(1-z)^{\frac{1}{2}}\right] = {}_2F_1\left[a, b; a+b+\frac{1}{2}; z\right] . \qquad (2.4)$$

Let $(1-z)^{\frac{1}{2}} = x$, $a = -n/2$, $b = n/2 + \alpha + \frac{1}{2}$, and $c = \alpha + 1 = a + b + \frac{1}{2}$ in (2.4). Then

$$ {}_2F_1\left[-n, n+2\alpha+1; \alpha+1; \frac{1}{2} - \frac{1}{2}x\right] = {}_2F_1\left[-\frac{n}{2}, \frac{n}{2} + \alpha + \frac{1}{2}; \alpha+1; 1-x^2\right] \qquad (2.5)$$

or, if we refer to (2.3) and replace n by $2n$ (2.5) becomes

$$P_{2n}^{(\alpha,\alpha)}(x) = \binom{2n+\alpha}{2n} \, {}_2F_1\left[-n, n+\alpha+\frac{1}{2}; \alpha+1; 1-x^2\right] \qquad (2.6)$$

and this is simply (2.2).

If

$$ {}_0F_1\left[\nu+1; \lambda\left(\frac{1+z}{2}\right)\right] = \sum_{n=0}^{\infty} A_n P_n^{\left(\alpha, -\frac{1}{2}\right)}(z) , \qquad (2.7)$$

then

$$A_n = h_n^{-1} \int_{-1}^{1} {}_0F_1\left[\nu+1; \lambda\left(\frac{1+z}{2}\right)\right] (1-z)^{\alpha} (1+z)^{-\frac{1}{2}} P_n^{\left(\alpha, -\frac{1}{2}\right)}(z) , \qquad (2.8)$$

or, equivalently,

$$A_n = (h_n 2^n n!)^{-1} \int_{-1}^{1} \frac{d^n}{dz^n}\, {}_0F_1\left[\nu+1;\ \frac{\lambda(1+z)}{2}\right](1-z)^{\alpha+n}(1+z)^{n-\frac{1}{2}}\, dz \quad , \quad (2.9)$$

$$= \frac{(h_n 2^n n!)^{-1}}{(\nu+1)_n}\left(\frac{\lambda}{2}\right)^n \int_{-1}^{1} {}_0F_1\left[\nu+n+1; \frac{\lambda}{2}(1+z)\right](1-z)^{\alpha+n}(1+z)^{n-\frac{1}{2}}\, dz \quad , \quad (2.10)$$

and

$$h_n = \frac{2^{\alpha+\frac{1}{2}}\,(n+1)_\alpha}{\left(2n+\alpha+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)_\alpha} \quad . \qquad (2.11)$$

Now expand the hypergeometric function in (2.10) in its power series and integrate term by term to get

$$A_n = \frac{\lambda^n}{\left(n+\alpha+\frac{1}{2}\right)_n (\nu+1)_n}\, {}_1F_2\left[n+\frac{1}{2}\ ;\ \nu+n+1, 2n+\alpha+\frac{3}{2}\ ;\ \lambda\right] \quad . \qquad (2.12)$$

Let $z = (2x^2-1)$ in (2.7) and $\lambda = -a^2/4$; and use (2.2). Then

$$J_\nu(ax) = \left(\frac{ax}{2}\right)^\nu \sum_{n=0}^{\infty} B_n P_{2n}^{(\alpha,\alpha)}(x) \ , \quad -1 \le x \le 1 \qquad (2.13)$$

where

$$B_n = \frac{(-)^n (2a)^{2n}}{\sqrt{\pi}\,(2n+2\alpha+1)_{2n}\left(n+\frac{1}{2}\right)_{\nu+\frac{1}{2}}}\, {}_1F_2\left[n+\frac{1}{2}\ ;\ \nu+n+1, 2n+\alpha+\frac{3}{2};\ -\frac{a^2}{4}\right] . \ (2.14)$$

For the Chebyshev case $\alpha = -\frac{1}{2}$ and

$$J_{\nu}(ax) = \left(\frac{ax}{2}\right)^{\nu} \sum_{n=0}^{\infty} C_n T_{2n}(x) \,, \quad -1 \leq x \leq 1 \quad, \tag{2.15}$$

where

$$C_n = \frac{\epsilon_n (-)^n (a/4)^{2n}}{n!\, \Gamma(\nu+n+1)} \, {}_1F_2 \left[n+\frac{1}{2} \, ; \nu+n+1, 2n+1; -\frac{a^2}{4} \right] \quad . \tag{2.16}$$

Notice that when $\nu = -\frac{1}{2}$, (2.14) simplifies. Also, since

$$J_{-\frac{1}{2}}(ax) = \left(\frac{\pi ax}{2}\right)^{-\frac{1}{2}} \cos (ax) \tag{2.17}$$

we infer the expansion

$$\cos (ax) = \sum_{n=0}^{\infty} C_n P_{2n}^{(\alpha,\alpha)}(x) \tag{2.18}$$

where

$$C_n = \frac{(-)^n \pi^{\frac{1}{2}} 2^{-\alpha+\frac{1}{2}} \left(2n+\alpha+\frac{1}{2}\right)(2n+\alpha+1)_{\alpha}}{a^{\alpha+\frac{1}{2}}} J_{2n+\alpha+\frac{1}{2}}(a) \quad , \tag{2.19}$$

a formula which can be derived in a number of different ways.

Another interesting and quite rapidly convergent expression for the coefficients in (2.13) can be derived by substituting the expansion (see Watson [13, p. 142])

$$J_\nu(ax)\left(\frac{ax}{2}\right)^{-\nu} = \left(\frac{2}{a}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{a}{2}\right)^k J_{\nu+k}(a)(1-x^2)^k}{k!} \quad , \qquad (2.20)$$

for $f(x)$ in (1.3). Employing a known integral formula [15], we have

$$E_n = \frac{2^{\nu-2\alpha-1}\sqrt{\pi}(4n+2\alpha+1)(2n+\alpha+1)_\alpha}{a^\nu (2n+1)_\alpha} \sum_{k=0}^{\infty} \frac{\Phi_{n,k}\left(\frac{a}{2}\right)^k J_{\nu+k}(a)}{(\alpha+k+1)_{\frac{1}{2}}} \quad , \qquad (2.21)$$

where

$$\Phi_{n,k} = {}_3F_2(-2n, 2n+2\alpha+1, \alpha+k+1; \alpha+1, 2k+2\alpha+2; 1) \quad . \qquad (2.22)$$

If the expansion (2.13) is truncated after N terms, we can obtain rough error estimates for large N by assuming that the expansion converges rapidly enough so that the first neglected term is approximately the error incurred. Using the relation

$$\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} = n^{\alpha-\beta}\left[1+0\left(\frac{1}{n}\right)\right] \quad , \qquad (2.23)$$

we see that the ${}_1F_2$ in (2.14) may be replaced by an exponential term. Also

$$\left|P_{2n}^{(\alpha,\alpha)}(x)\right| \le \binom{2n+\alpha}{2n}, \quad -1 \le x \le 1 \quad , \qquad (2.24)$$

so for (2.13) we have roughly

$$|\epsilon_N| \sim \frac{|a|^{\nu+2N}|x|^\nu\sqrt{\pi}\left|e^{-\frac{a^2}{8N}}\right|\Gamma(2N+2\alpha+1)}{2^{4N+2\alpha+\nu}\,\Gamma\left(2N+\alpha+\frac{1}{2}\right)\Gamma(N+\nu+1)\,\Gamma(\alpha+1)} \quad , \quad -1 \le x \le 1 \quad . \qquad (2.25)$$

3. Expansions of Bessel Functions of the Second Kind

The Bessel function and modified Bessel function of the second kind are denoted by $Y_\nu(z)$ and $K_\nu(z)$, respectively, and a treatment of them can be found in [2, v. 2, Ch. VII].* If ν is nonintegral, then

$$Y_\nu(z) = \left[\sin(\nu\pi)\right]^{-1}\left\{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)\right\} \quad , \qquad (3.1)$$

and

$$K_\nu(z) = \frac{\pi}{2}\left[\sin(\nu\pi)\right]^{-1}\left\{I_{-\nu}(z) - I_\nu(z)\right\} \quad , \qquad (3.2)$$

so for such values of ν expansions for the functions follow directly from the results of section 2.

If ν is an integer, it can be shown that

$$Y_k(ax) = \frac{2}{\pi} J_k(ax)\log\left(\frac{ax}{2}\right) + N_{k-1}(ax) - \frac{1}{\pi} G_k(ax) \quad , \qquad (3.3)$$

and

$$K_k(ax) = (-)^{k+1} I_k(ax)\log\left(\frac{ax}{2}\right) - \frac{\pi}{2} i^k N_{k-1}(iax) + \frac{i^k}{2} G_k(iax) \quad , \qquad (3.4)$$

where

$$N_{k-1}(ax) = \begin{cases} -\dfrac{1}{\pi}\displaystyle\sum_{m=0}^{k-1}\left(\frac{ax}{2}\right)^{2m-k}\frac{(k-m-1)!}{m!} , & k>0 \\ 0 , & k = 0 , \end{cases} \qquad (3.5)$$

* This reference calls $K_\nu(z)$ the modified Bessel function of the third kind.

and

$$G_k(ax) = \sum_{m=0}^{\infty} (-)^n \left(\frac{ax}{2}\right)^{k+2m} \frac{\left[\psi(k+m+1)+\psi(m+1)\right]}{m!\,(k+m)!} \quad . \tag{3.6}$$

We assume the value of $\log\left(\frac{ax}{2}\right)$ is known. Then, since expansions for $J_k(ax)$ and $I_k(ax)$ were found in section 1, and since $N_{k-1}(ax)$ is simply a polynomial in $1/(ax)$, we need expand only the entire part of (3.3), i.e., $G_k(ax)$, in symmetric Jacobi polynomials.

Using the representation (3.6) as $f(x)$ in formula (1.5), a straightforward derivation gives the series

$$G_k(ax) = \sum_{n=0}^{\infty} A_n P_n^{(\alpha,\alpha)}(x) \, , \quad -1 \leq x \leq 1 \quad , \tag{3.7}$$

where

$$A_n = \frac{\left[(-)^k+(-)^n\right](n+\alpha+1)_\alpha \left(n+\alpha+\frac{1}{2}\right)}{2^{n+2\alpha+1}} \sum_{m=0}^{\infty} \frac{(-)^m(-k-2m)_n}{\left(m+\frac{k-n+1}{2}\right)_{n+\alpha+1}}$$

$$\times \left(\frac{a}{2}\right)^{k+2m} \frac{\left[\psi(k+m+1)+\psi(m+1)\right]}{m!\,(k+m)!} \quad . \tag{3.8}$$

4. Conclusion

The symmetric Jacobi expansions given in this paper converge, in general, most rapidly when $\alpha = -\frac{1}{2}$, i.e., for the Chebyshev case.

Another advantage gained when $\alpha = -\frac{1}{2}$ is that the hypergeometric functions occurring in the coefficients of the expansions in several cases simplify considerably.

By using the same techniques expounded in this paper, expansions of Bessel functions in Laguerre, and hence by a simple transformation, Hermite polynomials can be obtained, but the convergence of these expansions is vastly inferior to that of those derived here.

Since our expansions converge much more rapidly than the corresponding Taylor's series representations, they will be of great utility as subroutines for evaluating Bessel functions on digital computers. We emphasize the need for calculating in the near future the coefficients of these expansions, particularly for the Chebyshev cases.

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Midwest Research Institute, Kansas City, Mo. POLYNOMIAL EXPANSIONS OF BESSEL FUNCTIONS AND SOME ASSOCIATED FUNCTIONS, by J. Jet Wimp November 1960. 15p. incl. 15 refs. (Proj. 7062; Task 70924) WADD TR 60-902 (Contract No. AF 33(616)-6535)

Unclassified Report

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Midwest Research Institute, Kansas City, Mo. POLYNOMIAL EXPANSIONS OF BESSEL FUNCTIONS AND SOME ASSOCIATED FUNCTIONS, by J. Jet Wimp November 1960. 15p. incl. 15 refs. (Proj. 7062; Task 70924) WADD TR 60-902 (Contract No. AF 33(616)-6535)

Unclassified Report

In this report we first determine representations for the Anger-Weber functions $\mathbf{J}_\nu(ax)$ and $\mathbf{E}_\nu(ax)$ in series of symmetric Jacobi polynomials. (These include Legendre and Chebyshev polynomials as special cases.) If

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1. Approximate Computation - Theory
2. Bessel Functions
3. Hypergeometric Functions
4. Polynomials - Applications
5. Series Expansions-Applications

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